# Exercise 2

Solve the given ODEs:

$$y'' + 4y = 4x, \; y(0) = 0, \; y'(0) = 1$$

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## Solution

This is a linear ODE with two initial conditions, so the Laplace transform can be used to solve the problem. The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx,$$

so the derivatives of $f(x)$ transform as follows.

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(x)\} = s^2 F(s) - sf(0) - f'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{4x\}$$

Use the fact that the operator is linear.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 4\mathcal{L}\{x\}$$

Use the expressions above for the transforms of the derivatives and use the definition on the right side.

$$s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = 4\int_0^\infty xe^{-sx}\,dx$$

Factor $Y(s)$ and solve the integral with integration by parts.

$$(s^2+4)Y(s) - sy(0) - y'(0) = 4\left[\frac{x}{(-s)}e^{-sx} - \frac{1}{(-s)^2}e^{-sx}\right]\Bigg|_0^\infty$$

Here we use the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$(s^2+4)Y(s) - 1 = \frac{4}{s^2}$$

Solve the equation for $Y(s)$.

$$(s^2+4)Y(s) = 1 + \frac{4}{s^2}$$
$$(s^2+4)Y(s) = \frac{s^2+4}{s^2}$$
$$Y(s) = \frac{1}{s^2}$$

Now that we have $Y(s)$, we can obtain $y(x)$ by taking the inverse Laplace transform of it.

$$y(x) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

Therefore,

$$y(x) = x.$$